

Carrier phase and amplitude estimation for phase shift keying using pilots and data

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Abstract—We consider least squares estimators of carrier phase and amplitude from a noisy communications signal that contains both pilot signals, known to the receiver, and data signals, unknown to the receiver. We focus on signaling constellations that have symbols evenly distributed on the complex unit circle, i.e., M -ary phase shift keying. We show, under reasonably mild conditions on the distribution of the noise, that the least squares estimator of carrier phase is strongly consistent and asymptotically normally distributed. However, the amplitude estimator is not consistent, but converges to a positive real number that is a function of the true carrier amplitude, the noise distribution and the size of the constellation. Our theoretical results can also be applied to the case where no pilot symbols exist, i.e., noncoherent detection. The results of Monte Carlo simulations are provided and these agree with the theoretical results.

Index Terms—Detection, phase shift keying, asymptotic statistics

I. INTRODUCTION

In passband communication systems the transmitted signal typically undergoes time offset (delay), phase shift and attenuation (amplitude change). These effects must be compensated for at the receiver. In this paper we assume that the time offset has been previously handled, and we focus on estimating the phase shift and attenuation. We consider signalling constellations that have symbols evenly distributed on the complex unit circle such as binary phase shift keying (BPSK), quaternary phase shift keying (QPSK) and M -ary phase shift keying (M -PSK). In this case, the transmitted symbols take the form,

$$s_i = e^{ju_i},$$

where $j = \sqrt{-1}$ and u_i is from the set $\{0, \frac{2\pi}{M}, \dots, \frac{2\pi(M-1)}{M}\}$ and $M \geq 2$ is the size of the constellation. We assume that some of the transmitted symbols are *pilot symbols* known to the receiver and the remainder are information carrying *data symbols* with phase that is unknown to the receiver. So,

$$s_i = \begin{cases} p_i & i \in P \\ d_i & i \in D, \end{cases}$$

where P is the set of indices describing the position of the pilot symbols p_i , and D is a set of indices describing the position

of the data symbols d_i . The sets P and D are disjoint, i.e., $P \cap D = \emptyset$ where \emptyset is the empty set, and we let $L = |P \cup D|$ be the total number of symbols transmitted.

We assume that time offset estimation has been performed and that L noisy M -PSK symbols are observed by the receiver. The received signal after matched filtering is,

$$y_i = a_0 s_i + w_i, \quad i \in P \cup D, \quad (1)$$

where w_i is noise and $a_0 = \rho_0 e^{j\theta_0}$ is a complex number representing both carrier phase θ_0 and amplitude ρ_0 (by definition ρ_0 is a positive real number). Our aim is to estimate a_0 from the noisy symbols $\{y_i, i \in P \cup D\}$. Complicating matters is that the data symbols $\{d_i, i \in D\}$ are not known to the receiver and must also be estimated. Estimation problems of this type have undergone extensive prior study [2–10]. A practical approach is the least squares estimator, that is, the minimisers of the sum of squares function

$$\begin{aligned} SS(a, \{d_i, i \in D\}) &= \sum_{i \in P \cup D} |y_i - a s_i|^2 \\ &= \sum_{i \in P} |y_i - a p_i|^2 + \sum_{i \in D} |y_i - a d_i|^2, \end{aligned} \quad (2)$$

where $|x|$ denotes the magnitude of the complex number x . The least squares estimator is also the maximum likelihood estimator under the assumption that the noise sequence $\{w_i, i \in \mathbb{Z}\}$ is additive white and Gaussian. However, as we show, the estimator works well under less stringent assumptions.

The existing literature [2–8] mostly considers what is called *noncoherent detection* where no pilot symbols exist ($P = \emptyset$). In the noncoherent setting *differential encoding* is often used, and for this reason the estimation problem has been called *multiple symbol differential detection*. A popular approach is the so called *non-data aided*, sometimes also called *non-decision directed*, estimator based on the paper of Viterbi and Viterbi [2]. The idea is to ‘strip’ the modulation from the received signal by taking $y_i/|y_i|$ to the power of M . A function $F: \mathbb{R} \mapsto \mathbb{R}$ is chosen and the estimator of the carrier phase θ_0 is taken to be $\frac{1}{M} \angle A$ where \angle denotes the complex argument and

$$A = \frac{1}{L} \sum_{i \in P \cup D} F(|y_i|) \left(\frac{y_i}{|y_i|} \right)^M. \quad (3)$$

Various choices for F are suggested in [2] and a statistical analysis is presented. A caveat of this estimator is that it is not obvious how pilot symbols should be included. This problem does not occur with the least square estimator.

An important paper is by Mackenthun [7] who described an algorithm to compute the least squares estimator requiring

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only $O(L \log L)$ arithmetic operations. Sweldens [8] rediscovered Mackenthun's algorithm in 2001. Both Mackenthun and Sweldens considered only the noncoherent setting, but we show in Section II that Mackenthun's algorithm can be modified to include pilot symbols. Our model includes the noncoherent case by setting the number of pilot symbols to zero, that is, putting $P = \emptyset$.

In the literature it has been common to assume that the data symbols $\{d_i, i \in D\}$ are of primary interest and that the complex amplitude a_0 is a nuisance parameter. The metric of performance is correspondingly the *symbol error rate*, or *bit error rate*. While estimating the symbols (or more precisely the transmitted bits) is ultimately the goal, we take the opposite point of view here. Our aim is to estimate a_0 , and we treat the unknown data symbols as nuisance parameters. This is motivated by the fact that in many modern communication systems the data symbols are *coded*. For this reason raw symbol error rate is not of interest at this stage. Instead, we desire an accurate estimator \hat{a} of a_0 , so that the compensated received symbols $\hat{a}^{-1}y_i$ can be accurately modelled using an additive noise channel. The additive noise channel is a common assumption for subsequent receiver operations, such as decoding. The estimator \hat{a} is also used in the computation of decoder metrics for modern decoders, and for interference cancellation in multiuser systems. Consequently, our metric of performance will not be symbol or bit error rate, but $|\hat{a} - a_0|$. It will be informative to consider the carrier phase and amplitude estimators separately, that is, if $\hat{a} = \hat{\rho}e^{j\hat{\theta}}$ where $\hat{\rho}$ is a positive real number, then we consider $|\langle \hat{\theta} - \theta_0 \rangle_\pi|$ and $|\hat{\rho} - \rho_0|$. The function $\langle \cdot \rangle_\pi$ denotes its argument taken 'modulo 2π ' into the interval $[-\pi, \pi)$. It will become apparent why $\langle \hat{\theta} - \theta_0 \rangle_\pi$ rather than $\hat{\theta} - \theta_0$ is the appropriate measure of error for the phase parameter.

It is possible to generalise the results we present here to allow data symbols with varying constellation size, i.e. varying M . For example, one might give more importance to certain data symbols and use BPSK ($M = 2$) for these, but QPSK ($M = 4$) for other less important symbols. This is related to what is called *unequal error protection* in the literature [11, 12]. To keep our ideas and notation focused we don't consider this further here.

The paper is organised in the following way. Section II extends Mackenthun's algorithm for the coherent case, when both pilot symbols and data symbols exist. Section III describes properties of complex random variables that we need. Section IV states two theorems that describe the statistical properties of the least squares estimator of carrier phase $\hat{\theta}$ and amplitude $\hat{\rho}$. We show, under some reasonably general assumptions about the distribution of the noise w_1, \dots, w_L , that $\langle \hat{\theta} - \theta_0 \rangle_\pi$ converges almost surely to zero and that $\sqrt{L}(\hat{\theta} - \theta_0)_\pi$ is asymptotically normally distributed as $L \rightarrow \infty$. However, $\hat{\rho}$ is not a consistent estimator of the amplitude ρ_0 . The asymptotic bias of $\hat{\rho}$ is small when the signal to noise ratio (SNR) is large, but the asymptotic bias is significant when the SNR is small. Sections V and VI provide proofs of the theorems stated in Section IV. In Section VII we consider the special case when the noise is Gaussian. In this case, our expressions for the asymptotic distribution can

be simplified. Section VIII presents the results of Monte-Carlo simulations. These simulations agree with the derived asymptotic properties.

II. MACKENTHUN'S ALGORITHM WITH PILOTS

In this section we derive Mackenthun's algorithm to compute the least squares estimator of the carrier phase and amplitude [7]. Mackenthun specifically considered the noncoherent setting, so we modify the algorithm to include the pilot symbols. For the purpose of analysing computational complexity, we will assume that the number of data symbols $|D|$ is proportional to the total number of symbols L , so that, for example, $O(L) = O(|D|)$. In this case Mackenthun's algorithm requires $O(L \log L)$ arithmetic operations. This complexity arises from the need to sort a list of $|D|$ elements.

Define the sum of squares function

$$\begin{aligned} SS(a, \{d_i, i \in D\}) &= \sum_{i \in P \cup D} |y_i - as_i|^2 \\ &= \sum_{i \in P \cup D} (|y_i|^2 - as_i y_i^* - a^* s_i^* y_i + aa^*), \end{aligned} \quad (4)$$

where $*$ denotes complex conjugate. The minimiser of SS with respect to a as a function of $\{d_i, i \in D\}$ is

$$\hat{a}(\{d_i, i \in D\}) = \frac{1}{L} \sum_{i \in P \cup D} y_i s_i^* = \frac{1}{L} Y \quad (5)$$

where $L = |P \cup D|$ is the total number of symbols transmitted, and to simplify our notation we have put

$$Y = \sum_{i \in P \cup D} y_i s_i^* = \sum_{i \in P} y_i p_i^* + \sum_{i \in D} y_i d_i^*.$$

Note that Y is a function of the unknown data symbols $\{d_i, i \in D\}$ and we could write $Y(\{d_i, i \in D\})$, but have chosen to suppress the argument $(\{d_i, i \in D\})$ for notational brevity. Substituting $\frac{1}{L}Y$ for a into (4) we obtain SS minimised with respect to a ,

$$SS(\{d_i, i \in D\}) = A - \frac{1}{L}|Y|^2, \quad (6)$$

where $A = \sum_{i \in P \cup D} |y_i|^2$ does not depend on the d_i . The least squares estimators of the data symbols are the minimisers of (6). Observe that given candidate values for the data symbols, we can compute the corresponding $SS(\{d_i, i \in D\})$ in $O(L)$ arithmetic operations. It turns out that there are at most $M|D|$ candidate values of the least squares estimator of the data symbols [7, 8].

To see this, let $a = \rho e^{j\theta}$ where ρ is a nonnegative real. Now,

$$\begin{aligned} SS(\rho, \theta, \{d_i, i \in D\}) &= \sum_{i \in P \cup D} |y_i - \rho e^{j\theta} s_i|^2 \\ &= \sum_{i \in P} |y_i - \rho e^{j\theta} p_i|^2 + \sum_{i \in D} |y_i - \rho e^{j\theta} d_i|^2. \end{aligned} \quad (7)$$

We have slightly abused notation here by reusing SS . This should not cause confusion as $SS(a, \{d_i, i \in D\})$, $SS(\rho, \theta, \{d_i, i \in D\})$, and $SS(\{d_i, i \in D\})$ are easily told apart by their arguments. For given θ , the least squares

estimator of the i th data symbol d_i is given by minimising $|y_i - \rho e^{j\theta} d_i|^2$, that is,

$$\hat{d}_i(\theta) = e^{j\hat{u}_i(\theta)} \quad \text{where} \quad \hat{u}_i(\theta) = \lfloor \angle(e^{-j\theta} y_i) \rfloor, \quad (8)$$

where $\angle(\cdot)$ denotes the complex argument (or phase), and $\lfloor \cdot \rfloor$ rounds its argument to the nearest multiple of $\frac{2\pi}{M}$. A word of caution, the notation $\lfloor \cdot \rfloor$ is often used to denote rounding to the nearest *integer*. This is not the case here. If the function $\text{round}(\cdot)$ takes its argument to the nearest integer then,

$$\lfloor x \rfloor = \frac{2\pi}{M} \text{round} \left(\frac{M}{2\pi} x \right).$$

Note that $\hat{d}_i(\theta)$ does not depend on ρ . As defined, $\hat{u}_i(\theta)$ is not strictly inside the set $\{0, \frac{2\pi}{M}, \dots, \frac{2\pi(M-1)}{M}\}$, but this is not of consequence, as we intend its value to be considered equivalent modulo 2π . With this in mind,

$$\hat{u}_i(\theta) = \lfloor \angle y_i - \theta \rfloor$$

which is equivalent to the definition from (8) modulo 2π .

We only require to consider θ in the interval $[0, 2\pi)$. Consider how $\hat{d}_i(\theta)$ changes as θ varies from 0 to 2π . Let $b_i = \hat{d}_i(0)$ and let

$$z_i = \angle y_i - \hat{u}_i(0) = \angle y_i - \lfloor \angle y_i \rfloor$$

be the phase difference between the received symbol y_i and the hard decision resulting when $\theta = 0$, i.e. $\lfloor \angle y_i \rfloor$. Then,

$$\hat{d}_i(\theta) = \begin{cases} b_i, & 0 \leq \theta < z_i + \frac{\pi}{M} \\ b_i e^{-j2\pi/M}, & z_i + \frac{\pi}{M} \leq \theta < z_i + \frac{3\pi}{M} \\ \vdots & \\ b_i e^{-j2\pi k/M}, & z_i + \frac{\pi(2k-1)}{M} \leq \theta < z_i + \frac{\pi(2k+1)}{M} \\ \vdots & \\ b_i e^{-j2\pi} = b_i, & z_i + \frac{\pi(2M-1)}{M} \leq \theta < 2\pi. \end{cases} \quad (9)$$

Let

$$f(\theta) = \{\hat{d}_i(\theta), i \in D\}$$

be a function mapping the interval $[0, 2\pi)$ to a sequence of M -PSK symbols indexed by the elements of D . Observe that $f(\theta)$ is piecewise continuous. The subintervals of $[0, 2\pi)$ over which $f(\theta)$ remains constant are determined by the values of $\{z_i, i \in D\}$. Let

$$S = \{f(\theta) \mid \theta \in [0, 2\pi)\}$$

be the set of all sequences $f(\theta)$ as θ varies from 0 to 2π . If $\hat{\theta}$ is the least squares estimator of the phase then S contains the sequence $\{\hat{d}_i(\hat{\theta}), i \in D\}$ corresponding to the least squares estimator of the data symbols, i.e., S contains the minimiser of (6). Observe from (9) that there are at most $M|D|$ sequences in S , because there are M distinct values of $d_i(\theta)$ for each $i \in D$ as θ varies from 0 to 2π .

The sequences in S can be enumerated as follows. Let σ denote the permutation of the indices in D such that $z_{\sigma(i)}$ are in ascending order, that is,

$$z_{\sigma(i)} \leq z_{\sigma(k)} \quad (10)$$

whenever $i < k$ where $i, k \in \{0, 1, \dots, |D| - 1\}$. It is convenient to define the indices into σ to be taken modulo $|D|$, that is, if m is an integer not from $\{0, 1, \dots, |D| - 1\}$ then we define $\sigma(m) = \sigma(k)$ where $k \equiv m \pmod{|D|}$ and $k \in \{0, 1, \dots, |D| - 1\}$. The first sequence in S is

$$f_0 = f(0) = \{\hat{d}_i(0), i \in D\} = \{b_i, i \in D\}.$$

The next sequence f_1 is given by replacing the element $b_{\sigma(0)}$ in f_0 with $b_{\sigma(0)} e^{-j2\pi/M}$. Given a sequence x we use $x e_i$ to denote x with the i th element replaced by $x_i e^{-j2\pi/M}$. Using this notation,

$$f_1 = f_0 e_{\sigma(0)}.$$

The next sequence in S is correspondingly

$$f_2 = f_0 e_{\sigma(0)} e_{\sigma(1)} = f_1 e_{\sigma(1)},$$

and the k th sequence is

$$f_{k+1} = f_k e_{\sigma(k)}. \quad (11)$$

In this way, all $M|D|$ sequences in S can be recursively enumerated.

We want to find the $f_k \in S$ corresponding to the minimiser of (6). A naïve approach would be to compute $SS(f_k)$ for each $k \in \{0, 1, \dots, M|D| - 1\}$. Computing $SS(f_k)$ for any particular k requires $O(L)$ arithmetic operations. So, this naïve approach would require $O(LM|D|) = O(L^2)$ operations in total. Following Mackenthun [7], we show how $SS(f_k)$ can be computed recursively.

Let,

$$SS(f_k) = A - \frac{1}{L} |Y_k|^2, \quad (12)$$

where,

$$\begin{aligned} Y_k &= Y(f_k) = \sum_{i \in P} y_i p_i^* + \sum_{i \in D} y_i f_{ki}^* \\ &= B + \sum_{i \in D} g_{ki}, \end{aligned}$$

where $B = \sum_{i \in P} y_i p_i^*$ is independent of the data symbols, and f_{ki} denotes the i th symbol in f_k , and for convenience, we put $g_{ki} = y_i f_{ki}^*$. Letting g_k be the sequence $\{g_{ki}, i \in D\}$ we have, from (11), that g_k satisfies the recursive equation

$$g_{k+1} = g_k e_{\sigma(k)}^*,$$

where $g_k e_{\sigma(k)}^*$ indicates the sequence g_k with the $\sigma(k)$ th element replaced by $g_{k\sigma(k)} e^{j2\pi/M}$. Now,

$$Y_0 = B + \sum_{i \in D} g_{0i}$$

can be computed in $O(L)$ operations, and

$$\begin{aligned} Y_1 &= B + \sum_{i \in D} g_{1i} \\ &= B + (e^{j2\pi/M} - 1) g_{0\sigma(0)} + \sum_{i \in D} g_{0i} \\ &= Y_0 + \eta g_{0\sigma(0)}, \end{aligned}$$

where $\eta = e^{j2\pi/M} - 1$. In general,

$$Y_{k+1} = Y_k + \eta g_{k\sigma(k)}.$$

Input: $\{y_i, i \in P \cup D\}$

```

1 for  $i \in D$  do
2    $\phi = \angle y_i$ 
3    $u = \lfloor \phi \rfloor$ 
4    $z_i = \phi - u$ 
5    $g_i = y_i e^{-ju}$ 
6  $Y = \sum_{i \in P} y_i p_i^* + \sum_{i \in D} g_i$ 
7  $\hat{a} = \frac{1}{L} Y$ 
8  $\hat{Q} = \frac{1}{L} |Y|^2$ 
9  $\eta = e^{j2\pi/M} - 1$ 
10  $\sigma = \text{sortindices}(z)$ 
11 for  $k = 0$  to  $M|D| - 1$  do
12    $Y = Y + \eta g_{\sigma(k)}$ 
13    $g_{\sigma(k)} = (\eta + 1) g_{\sigma(k)}$ 
14    $Q = \frac{1}{L} |Y|^2$ 
15   if  $Q > \hat{Q}$  then
16      $\hat{Q} = Q$ 
17      $\hat{a} = \frac{1}{L} Y$ 
18 return  $\hat{a}$ 

```

Algorithm 1: Mackenthun's algorithm with pilot symbols

So, each Y_k can be computed from its predecessor Y_{k-1} in a $O(1)$ arithmetic operations. Given Y_k , the value of $SS(f_k)$ can be computed in $O(1)$ operations using (12). Let $\hat{k} = \arg \min SS(f_k)$. The least squares estimator of a_0 is then computed according to (5),

$$\hat{a} = \frac{1}{L} Y_{\hat{k}}. \quad (13)$$

Pseudocode is given in Algorithm 1. Line 10 contains the function `sortindices` that, given $z = \{z_i, i \in D\}$, returns the permutation σ as described in (10). The `sortindices` function requires sorting $|D|$ elements. This requires $O(L \log L)$ operations. The `sortindices` function is the primary bottleneck in this algorithm when L is large. The loops on lines 1 and 11 and the operations on lines 6 to lines 8 all require $O(L)$ or less operations.

III. CIRCULARLY SYMMETRIC COMPLEX RANDOM VARIABLES

Before describing the statistical properties of the least squares estimator, we first require some properties of complex valued random variables. A complex random variable W is said to be *circularly symmetric* if its phase $\angle W$ is independent of its magnitude $|W|$ and if the distribution of $\angle W$ is uniform on $[0, 2\pi)$. That is, if $Z \geq 0$ and $\Theta \in [0, 2\pi)$ are real random variables such that $Z e^{j\Theta} = W$, then Θ is uniformly distributed on $[0, 2\pi)$ and is independent of Z . If the probability density function (pdf) of Z is $f_Z(z)$, then the joint pdf of Θ and Z is

$$f_{Z,\Theta}(z, \theta) = \frac{1}{2\pi} f_Z(z).$$

Observe that for any real number ϕ , the pdf of W and $e^{j\phi}W$ are the same, that is, the pdf is invariant to phase rotation. If

$\mathbb{E}|W| = \mathbb{E}Z$ is finite, then W has zero mean because

$$\begin{aligned} \mathbb{E}W &= \int_0^{2\pi} \int_0^\infty z e^{j\theta} f_{Z,\Theta}(z, \theta) dz d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{j\theta} \int_0^\infty z f_Z(z) dz d\theta \\ &= \frac{1}{2\pi} \mathbb{E}Z \int_0^{2\pi} e^{j\theta} d\theta = 0. \end{aligned}$$

If X and Y are real random variables equal to the real and imaginary parts of $W = X + jY$ then the joint pdf of X and Y is

$$f_{X,Y}(x, y) = \frac{f_Z(\sqrt{x^2 + y^2})}{2\pi \sqrt{x^2 + y^2}}.$$

We will have particular use of complex random variables of the form $1 + W$ where W is circularly symmetric. Let $R \geq 0$ and $\Phi \in [0, 2\pi)$ be real random variables satisfying,

$$R e^{j\Phi} = 1 + W.$$

The joint pdf of R and Φ can be shown to be

$$f(r, \phi) = \frac{r f_Z(\sqrt{r^2 - 2r \cos \phi + 1})}{2\pi \sqrt{r^2 - 2r \cos \phi + 1}}. \quad (14)$$

Since $\cos \phi$ has period 2π and is even on $[-\pi, \pi]$ it follows that $f(r, \phi)$ has period 2π and is even on $[-\pi, \pi]$ with respect to ϕ . The mean of $R e^{j\Phi}$ is equal to one because the mean of W is zero. So,

$$\mathbb{E}\Re(R e^{j\Phi}) = \mathbb{E}R \cos(\Phi) = 1, \quad (15)$$

where $\Re(\cdot)$ denotes the real part, and

$$\mathbb{E}\Im(R e^{j\Phi}) = \mathbb{E}R \sin(\Phi) = 0, \quad (16)$$

where $\Im(\cdot)$ denotes the imaginary part.

IV. STATISTICAL PROPERTIES OF THE LEAST SQUARES ESTIMATOR

In this section we describe the asymptotic properties of the least squares estimator. In what follows we use $\langle x \rangle_\pi$ to denote x taken 'modulo 2π ' into the interval $[-\pi, \pi)$, that is

$$\langle x \rangle_\pi = x - 2\pi \text{round}\left(\frac{x}{2\pi}\right),$$

where $\text{round}(\cdot)$ takes its argument to the nearest integer. The direction of rounding for half-integers is not important so long as it is consistent. We have chosen to round up half-integers here. Similarly we use $\langle x \rangle$ to denote x taken 'modulo $\frac{2\pi}{M}$ ' into the interval $[-\frac{\pi}{M}, \frac{\pi}{M})$, that is

$$\langle x \rangle = x - \frac{2\pi}{M} \text{round}\left(\frac{M}{2\pi} x\right) = x - \lfloor x \rfloor.$$

The next two theorems describe the asymptotic properties of the least squares estimator. These are the central results and the chief original contributions of this paper.

Theorem 1. (Almost sure convergence) *Let $\{w_i\}$ be a sequence of independent and identically distributed, circularly symmetric complex random variables with $\mathbb{E}|w_1|^2$ finite, and let $\{y_i, i \in P \cup D\}$ be given by (1). Let $\hat{a} = \hat{\rho} e^{j\hat{\theta}}$ be the least*

squares estimator of $a_0 = \rho_0 e^{j\theta_0}$. Put $L = |P \cup D|$ and let $|P|$ and $|D|$ increase in such a way that

$$\frac{|P|}{L} \rightarrow p \quad \text{and} \quad \frac{|D|}{L} \rightarrow d \quad \text{as } L \rightarrow \infty.$$

Let $R_i \geq 0$ and $\Phi_i \in [0, 2\pi)$ be real random variables satisfying

$$R_i e^{j\Phi_i} = 1 + \frac{w_i}{a_0 s_i}, \quad (17)$$

and define the continuous function

$$G(x) = ph_1(x) + dh_2(x) \quad \text{where}$$

$$h_1(x) = \mathbb{E}R_1 \cos(x + \Phi_1), \quad h_2(x) = \mathbb{E}R_1 \cos(x + \Phi_1).$$

If $p > 0$ and if $G(x)$ is uniquely maximised at $x = 0$ over the interval $[-\pi, \pi]$ then

- 1) $\langle \hat{\theta} - \theta_0 \rangle_\pi \rightarrow 0$ almost surely as $L \rightarrow \infty$,
- 2) $\hat{\rho} \rightarrow \rho_0 G(0)$ almost surely as $L \rightarrow \infty$.

Theorem 2. (Asymptotic normality) Under the same conditions as Theorem 1, let $f(r, \phi)$ be the joint probability density function of R_1 and Φ_1 , let

$$g(\phi) = \int_0^\infty r f(r, \phi) dr$$

and assume that $\frac{|P|}{L} = p + o(L^{-1/2})$ and $\frac{|D|}{L} = d + o(L^{-1/2})$. Put $\hat{\lambda}_L = -\langle \hat{\theta} - \theta_0 \rangle_\pi = \langle \theta_0 - \hat{\theta} \rangle_\pi$ and $\hat{m}_L = \hat{\rho} - \rho_0 G(0)$. If the function g is continuous at $\frac{2\pi}{M}k + \frac{\pi}{M}$ for each $k = 0, \dots, M-1$, then the distribution of $(\sqrt{L}\hat{\lambda}_L, \sqrt{L}\hat{m}_L)$ converges to the bivariate normal with zero mean and covariance matrix

$$\begin{pmatrix} \frac{pA_1 + dA_2}{(p+Hd)^2} & 0 \\ 0 & \rho_0^2(pB_1 + dB_2) \end{pmatrix}$$

as $L \rightarrow \infty$, where

$$H = h_2(0) - 2 \sin\left(\frac{\pi}{M}\right) \sum_{k=0}^{M-1} g\left(\frac{2\pi}{M}k + \frac{\pi}{M}\right),$$

$$A_1 = \mathbb{E}R_1^2 \sin^2(\Phi_1), \quad A_2 = \mathbb{E}R_1^2 \sin^2 \langle \Phi_1 \rangle,$$

$$B_1 = \mathbb{E}R_1^2 \cos^2(\Phi_1) - 1, \quad B_2 = \mathbb{E}R_1^2 \cos^2 \langle \Phi_1 \rangle - h_2^2(0).$$

The proof of Theorem 1 is in Section V and the proof of Theorem 2 is in Section VI. Before giving the proofs we discuss the assumptions made by these theorems. The assumption that w_1, \dots, w_L are circularly symmetric can be relaxed, but this comes at the expense of making the theorem statements more complicated. If w_i is not circularly symmetric then the distribution of R_i and Φ_i may depend on a_0 and also on the transmitted symbols $\{s_i, i \in P \cup D\}$. As a result the asymptotic variance described in Theorem 2 depends on a_0 and $\{s_i, i \in P \cup D\}$, rather than just ρ_0 . The circularly symmetric assumption may not always hold in practice, but we feel it provides a sensible trade off between simplicity and generality.

The assumption that $\mathbb{E}|w_1|^2 = \mathbb{E}|w_i|^2$ is finite implies that R_i has finite variance since $\mathbb{E}R_i^2 = 1 + \mathbb{E}|w_i|^2$. This is required in Theorem 2 so that the constants A_1, A_2, B_1 and B_2 exist. We will also use the fact that R_i has finite variance to simplify

the proof of Theorem 1 by use of Kolmogorov's strong law of large numbers [13].

The theorems place conditions on $\langle \hat{\theta} - \theta_0 \rangle_\pi$ rather than directly on $\hat{\theta} - \theta_0$. This makes sense because the phases θ_0 and $\theta_0 + 2\pi k$ are equivalent for any integer k . So, for example, we expect the phases 0.99π and -0.99π to be close together, the difference between them being $|\langle -0.99\pi - 0.99\pi \rangle_\pi| = 0.02\pi$, and not $|-0.99\pi - 0.99\pi| = 1.98\pi$.

Theorem 2 requires the function g to be continuous at $\frac{2\pi}{M}k + \frac{\pi}{M}$ for each $k = 0, \dots, M-1$. This places mild restrictions on the distribution of the noise w_i . For example, the requirements are satisfied if the joint pdf of the real and imaginary parts of w_i is continuous, since in this case $f(r, \phi)$ is continuous. Because $f(r, \phi)$ has period 2π and is even on $[-\pi, \pi]$ with respect to ϕ it follows that g has period 2π and is even on $[-\pi, \pi]$.

A key assumption in Theorem 1 is that $G(x)$ is uniquely maximised at $x = 0$ for $x \in [-\pi, \pi]$. This assumption asserts that $G(x) \leq G(0)$ for all $x \in [-\pi, \pi]$ and that if $\{x_i\}$ is a sequence of numbers from $[-\pi, \pi]$ such that $G(x_i) \rightarrow G(0)$ as $i \rightarrow \infty$ then $x_i \rightarrow 0$ as $i \rightarrow \infty$. Although we will not prove it here, this assumption is not only sufficient, but also necessary, for if $G(x)$ is uniquely maximised at some $x \neq 0$ then $\langle \hat{\theta} - \theta_0 \rangle_\pi \rightarrow x$ almost surely as $L \rightarrow \infty$, while if $G(x)$ is not uniquely maximised then $\langle \hat{\theta} - \theta_0 \rangle_\pi$ will not converge. One can check that this assumption holds when w_1 is circularly symmetric and normally distributed. We will not attempt to further classify those distributions for which the assumption holds here.

Theorem 1 defines real numbers p and d to represent the proportion of pilot symbols and data symbols in the limit as L goes to infinity. For Theorem 2 we need the slightly stronger condition that

$$\frac{|P|}{L} = p + o(L^{-1/2}) \quad \text{and} \quad \frac{|D|}{L} = d + o(L^{-1/2}).$$

This stronger condition is required to prove the asymptotic normality of $\sqrt{L}\hat{m}_L$.

The next two sections give proofs of Theorems 1 and 2. The proofs make use of various lemmas, which are proved in the appendix.

V. PROOF OF ALMOST SURE CONVERGENCE (THEOREM 1)

Substituting $\{\hat{d}_i(\theta), i \in D\}$ from (8) into (7) we obtain SS minimised with respect to the data symbols,

$$\begin{aligned} SS(\rho, \theta) &= \sum_{i \in P} |y_i - \rho e^{j\theta} p_i|^2 + \sum_{i \in D} |y_i - \rho e^{j\theta} \hat{d}_i(\theta)|^2 \\ &= A - \rho Z(\theta) - \rho Z^*(\theta) + L\rho^2, \end{aligned}$$

where

$$Z(\theta) = \sum_{i \in P} y_i e^{-j\theta} p_i^* + \sum_{i \in D} y_i e^{-j\theta} \hat{d}_i^*(\theta),$$

and $Z^*(\theta)$ is the conjugate of $Z(\theta)$. Differentiating with respect to ρ and setting the resulting expression to zero gives the least squares estimator of ρ_0 as a function of θ ,

$$\hat{\rho}(\theta) = \frac{Z(\theta) + Z^*(\theta)}{2L} = \frac{1}{L} \Re(Z(\theta)), \quad (18)$$

where $\Re(\cdot)$ denotes the real part. Substituting this expression into $SS(\rho, \theta)$ gives SS minimised with respect to ρ and the data symbols,

$$SS(\theta) = A - \frac{1}{L} \Re(Z(\theta))^2.$$

We again abuse notation by reusing SS , but this should not cause confusion as $SS(\rho, \theta)$ and $SS(\theta)$ are easily told apart by their inputs. By definition the amplitude ρ_0 and its estimator $\hat{\rho}$ are positive. However, $\hat{\rho}(\theta) = \Re(Z(\theta))$ may take negative values for some $\theta \in [-\pi, \pi)$. The least square estimator $\hat{\theta}$ of θ_0 is the minimiser of $SS(\theta)$ under the constraint $\hat{\rho}(\theta) = \Re(Z(\theta)) > 0$. Equivalently $\hat{\theta}$ is the maximiser of $\Re(Z(\theta))$ with no constraints required.

We are thus interested in analysing the behaviour of the maximiser of $\Re(Z(\theta))$. Recalling the definition of R_i and Φ_i from (17),

$$\begin{aligned} y_i &= a_0 s_i + w_i \\ &= a_0 s_i \left(1 + \frac{w_i}{a_0 s_i} \right) \\ &= a_0 s_i R_i e^{j\Phi_i} \\ &= \rho_0 R_i e^{j(\Phi_i + \theta_0 + \angle s_i)}. \end{aligned}$$

Recalling the definition of $\hat{d}_i(\theta)$ and $\hat{u}_i(\theta)$ from (8),

$$\begin{aligned} \hat{u}_i(\theta) &= \lfloor \angle y_i - \theta \rfloor \\ &= \lfloor \theta_0 + \Phi_i + \angle s_i - \theta \rfloor \\ &\equiv \lfloor \langle \theta_0 - \theta \rangle_\pi + \Phi_i + \angle s_i \rfloor \pmod{2\pi} \\ &= \lfloor \lambda + \Phi_i + \angle s_i \rfloor, \end{aligned}$$

where we put $\lambda = \langle \theta_0 - \theta \rangle_\pi$ and where, as in Section II, we consider $\hat{u}_i(\theta)$ equivalent modulo 2π . Because $\hat{d}_i^*(\theta) = e^{-j\hat{u}_i(\theta)}$, it follows that, when $i \in D$,

$$\begin{aligned} y_i e^{-j\theta} \hat{d}_i^*(\theta) &= \rho_0 R_i e^{j(\lambda + \Phi_i + \angle s_i - \lfloor \lambda + \Phi_i + \angle s_i \rfloor)} \\ &= \rho_0 R_i e^{j(\lambda + \Phi_i - \lfloor \lambda + \Phi_i \rfloor)} \\ &= \rho_0 R_i e^{j\langle \lambda + \Phi_i \rangle} \end{aligned} \quad (19)$$

since $\lfloor x + \angle s_i \rfloor = \lfloor x \rfloor + \angle s_i$ for all $x \in \mathbb{R}$ as a result of $\angle s_i$ being a multiple of $\frac{2\pi}{M}$. Otherwise, when $i \in P$,

$$y_i e^{-j\theta} p_i^* = \rho_0 R_i e^{j(\lambda + \Phi_i)}.$$

Now,

$$Z(\theta) = \rho_0 \sum_{i \in P} R_i e^{j(\lambda + \Phi_i)} + \rho_0 \sum_{i \in D} R_i e^{j\langle \lambda + \Phi_i \rangle}.$$

Let

$$G_L(\lambda) = \frac{1}{\rho_0 L} \Re(Z(\theta)) \quad (20)$$

and put $\hat{\lambda}_L = -\langle \hat{\theta} - \theta_0 \rangle_\pi = \langle \theta_0 - \hat{\theta} \rangle_\pi$. Since $\hat{\theta}$ is the maximiser of $\Re(Z(\theta))$ it follows that $\hat{\lambda}_L$ is the maximiser of $G_L(\lambda)$. We will show that $\hat{\lambda}_L$ converges almost surely to zero as $L \rightarrow \infty$. The proof of part 1 of Theorem 1 follows from this.

Recall the functions G , h_1 and h_2 defined in the statement of Theorem 1. Observe that

$$\mathbb{E}G_L(\lambda) = \frac{|P|}{L} h_1(\lambda) + \frac{|D|}{L} h_2(\lambda)$$

and since $\frac{|P|}{L} \rightarrow p$ and $\frac{|D|}{L} \rightarrow d$ as $L \rightarrow \infty$,

$$\lim_{L \rightarrow \infty} \mathbb{E}G_L(\lambda) = G(\lambda) = p h_1(\lambda) + d h_2(\lambda).$$

As is customary, let Ω be the sample space on which the random variables $\{w_i\}$ are defined. Let A be the subset of the sample space Ω on which $G(\hat{\lambda}_L) \rightarrow G(0)$ as $L \rightarrow \infty$. Lemma 1 shows that $\Pr\{A\} = 1$. Let A' be the subset of the sample space on which $\hat{\lambda}_L \rightarrow 0$ as $L \rightarrow \infty$. Because $G(x)$ is uniquely maximised at $x = 0$, it follows that $G(\hat{\lambda}_L) \rightarrow G(0)$ only if $\hat{\lambda}_L \rightarrow 0$ as $L \rightarrow \infty$. So $A \subseteq A'$ and therefore $\Pr\{A'\} \geq \Pr\{A\} = 1$. Part 1 of Theorem 1 follows.

It remains to prove part 2 of the theorem regarding the convergence of the amplitude estimator $\hat{\rho}$. From (18),

$$\hat{\rho} = \frac{1}{L} \Re(Z(\hat{\theta})) = \rho_0 G_L(\hat{\lambda}_L). \quad (21)$$

Lemma 8 in the appendix shows that $G_L(\hat{\lambda}_L)$ converges almost surely to $G(0)$ as $L \rightarrow \infty$, and $\hat{\rho}$ consequently converges almost surely to $\rho_0 G(0)$ as required. It remains to prove Lemmas 1 and 8. These are proved in Section A of the appendix.

VI. PROOF OF ASYMPTOTIC NORMALITY (THEOREM 2)

We first prove the asymptotic normality of $\sqrt{L}\hat{\lambda}_L$. Once this is done we will be able to prove the normality of $\sqrt{L}\hat{m}_L$. Recall that $\hat{\lambda}_L$ is the maximiser of the function G_L defined in (20). The proof is complicated by the fact that G_L is not differentiable everywhere due to the function $\langle \cdot \rangle$ not being differentiable at multiples of $\frac{\pi}{M}$. This prevents the use of “standard approaches” to proving normality that are based on the mean value theorem [14–19]. However, Lemma 9 shows that the derivative G'_L does exist, and is equal to zero, at $\hat{\lambda}_L$. Similar properties have been used by some of the present authors to analyse the behaviour of polynomial-phase estimators [20]. Define the function

$$R_L(\lambda) = \frac{1}{L} \sum_{i \in P} R_i \sin(\lambda + \Phi_i) + \frac{1}{L} \sum_{i \in D} R_i \sin\langle \lambda + \Phi_i \rangle. \quad (22)$$

Whenever $G_L(\lambda)$ is differentiable $G'_L(\lambda) = R_L(\lambda)$, and so $R_L(\hat{\lambda}_L) = G'_L(\hat{\lambda}_L) = 0$ by Lemma 9. Let $Q_L(\lambda) = \mathbb{E}R_L(\lambda) - \mathbb{E}R_L(0)$ and write

$$\begin{aligned} 0 &= R_L(\hat{\lambda}_L) - Q_L(\hat{\lambda}_L) + Q_L(\hat{\lambda}_L) \\ &= \sqrt{L}(R_L(\hat{\lambda}_L) - Q_L(\hat{\lambda}_L)) + \sqrt{L}Q_L(\hat{\lambda}_L). \end{aligned}$$

Lemma 11 shows that

$$\sqrt{L}Q_L(\hat{\lambda}_L) = \sqrt{L}\hat{\lambda}_L(p + Hd + o_P(1))$$

where $o_P(1)$ denotes a sequence of random variables converging in probability to zero as $L \rightarrow \infty$, and p, d and H are defined in the statement of Theorems 1 and 2. Lemma 16 shows that

$$\sqrt{L}(R_L(\hat{\lambda}_L) - Q_L(\hat{\lambda}_L)) = o_P(1) + \sqrt{L}R_L(0).$$

It follows from the three equations above that,

$$0 = o_P(1) + \sqrt{L}R_L(0) + \sqrt{L}\hat{\lambda}_L(p + Hd + o_P(1))$$

and rearranging gives,

$$\sqrt{L}\hat{\lambda}_L = o_P(1) - \frac{\sqrt{L}R_L(0)}{p + Hd + o_P(1)}.$$

Lemma 17 shows that the distribution of $\sqrt{L}R_L(0)$ converges to the normal with zero mean and variance $pA_1 + dA_2$ where A_1 and A_2 are defined in the statement of Theorem 2. It follows that the distribution of $\sqrt{L}\hat{\lambda}_L$ converges to the normal with zero mean and variance

$$\frac{pA_1 + dA_2}{(p + Hd)^2}.$$

We now analyse the asymptotic distribution of $\sqrt{L}\hat{m}_L$. Let $T_L(\lambda) = \mathbb{E}G_L(\lambda)$. Using (21),

$$\begin{aligned}\sqrt{L}\hat{m}_L &= \sqrt{L}\rho_0(G_L(\hat{\lambda}_L) - G(0)) \\ &= \sqrt{L}\rho_0(G_L(\hat{\lambda}_L) - T_L(\hat{\lambda}_L) + T_L(\hat{\lambda}_L) - G(0)).\end{aligned}$$

Lemma 18 shows that

$$\sqrt{L}(G_L(\hat{\lambda}_L) - T_L(\hat{\lambda}_L)) = o_P(1) + X_L,$$

where $X_L = \sqrt{L}(G_L(0) - T_L(0))$. Lemma 19 shows that

$$\sqrt{L}(T_L(\hat{\lambda}_L) - G(0)) = o_P(1).$$

It follows that $\sqrt{L}\hat{m}_L = \rho_0 X_L + o_P(1)$. Lemma 20 shows that the distribution of X_L converges to the normal with zero mean and variance $pB_1 + dB_2$ as $L \rightarrow \infty$ where B_1 and B_2 are defined in the statement of Theorem 2. Thus, the distribution of $\sqrt{L}\hat{m}_L$ converges to the normal with zero mean and variance $\rho_0^2(pB_1 + dB_2)$ as required. Because X_L does not depend on $\hat{\lambda}_L$, it follows that $\text{cov}(X_L, \sqrt{L}\hat{\lambda}_L) = 0$, and so,

$$\text{cov}(\sqrt{L}\hat{m}_L, \sqrt{L}\hat{\lambda}_L) \rightarrow \text{cov}(\rho_0 X_L, \sqrt{L}\hat{\lambda}_L) = 0$$

as $L \rightarrow \infty$. The lemmas that we have used are proved in Section B of the appendix.

VII. THE GAUSSIAN NOISE CASE

Let the noise sequence $\{w_i\}$ be complex Gaussian with independent real and imaginary parts having zero mean and variance σ^2 . The joint density function of the real and imaginary parts is

$$\frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x^2 + y^2)}.$$

Theorems 1 and 2 hold, and since the distribution of w_1 is circularly symmetric, the distribution of $R_1 e^{j\Phi_1}$ is identical to the distribution of $1 + \frac{1}{\rho_0} w_1$. It can be shown that

$$g(\phi) = \frac{\cos \phi}{2\pi} e^{-\kappa} + \frac{\Psi(\sqrt{2\kappa} \cos \phi)}{\sqrt{\pi\kappa}} e^{-\kappa \sin^2 \phi} (2 + \kappa \cos^2 \phi)$$

where $\kappa = \frac{\rho_0^2}{2\sigma^2}$ and $\Psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} dx$ is the cumulative density function of the standard normal. The value of A_1, A_2, B_1 and B_2 can be efficiently computed by numerical integration using this formula.

VIII. SIMULATIONS

We present the results of Monte-Carlo simulations with the least squares estimator. In all simulations the noise samples w_1, \dots, w_L are independent and identically distributed circularly symmetric and Gaussian with real and imaginary parts having variance σ^2 . Under these conditions the least squares estimator is also the maximum likelihood estimator. Simulations are run with $M = 2, 4, 8$ (BPSK, QPSK, 8-PSK) and with signal to noise ratio $\text{SNR} = \frac{\rho_0^2}{2\sigma^2}$ between -20dB and 20dB in steps of 1dB. The amplitude $\rho_0 = 1$ and θ_0 is uniformly distributed on $[-\pi, \pi)$. For each value of SNR, $T = 5000$ replications are performed to obtain T estimates $\hat{\rho}_1, \dots, \hat{\rho}_T$ and $\hat{\theta}_1, \dots, \hat{\theta}_T$.

Figures 1, 2 and 3 show the sample mean square error (MSE) of the phase estimator when $M = 2, 4, 8$ with $L = 4096$ and for varying proportions of pilots symbols $|P| = 0, \frac{L}{32}, \frac{L}{8}, \frac{L}{2}, L$. When $|P| \neq 0$ (i.e. coherent detection) the mean square error is computed as $\frac{1}{T} \sum_{i=1}^T (\hat{\theta}_i - \theta_0)^2$. Otherwise, when $|P| = 0$ the mean square error is computed as $\frac{1}{T} \sum_{i=1}^T (\hat{\theta}_i - \theta_0)^2$ as in [1]. The dots, squares, circles and crosses are the results of Monte-Carlo simulations with the least square estimator. The solid lines are the estimator MSEs predicted by Theorem 2. The prediction is made by dividing the asymptotic covariance matrix by L . The theorem accurately predicts the behaviour of the phase estimator when L is sufficiently large. As the SNR decreases the variance of the phase estimator approaches that of the uniform distribution on $[-\pi, \pi)$ when $|P| \neq 0$ and the uniform distribution on $[-\frac{\pi}{M}, \frac{\pi}{M})$ when $|P| = 0$ [1]. Theorem 2 does not model this behaviour in the sense that for any fixed L there exist sufficiently small values of SNR for which Theorem 2 does not produce accurate predictions of the MSE. As the SNR increases the variance of the estimators converge to that of the estimator where all symbols are pilots, i.e. $|P| = L$.

Figures 1, 2 and 3 also display the sample MSE of the noncoherent phase estimator of Viterbi and Viterbi [2] described by (3). This estimator requires selection of a function F that transforms the amplitude of each sample prior to the final estimation step. Viterbi and Viterbi propose several viable alternatives, from which we have chosen $F(x) = 1$. The Viterbi and Viterbi estimator is only applicable in the noncoherent setting, i.e. when $|P| = 0$. The sample MSE of the least squares estimator (when $|P| = 0$) and the Viterbi and Viterbi estimator is similar. The least squares estimator appears slightly more accurate for some values of SNR.

Figures 4, 5 and 6 show the variance of the amplitude estimator when $M = 2, 4, 8$ and with $L = 32, 256, 2048$ and when the number of pilots symbols is $|P| = 0, \frac{L}{2}, L$. The solid lines are the variance predicted by Theorem 2. The dots and crosses show the results of Monte-Carlo simulations. Each point is computed as the unbiased error $\frac{1}{T} \sum_{i=1}^T (\hat{\rho}_i - \rho_0 G(0))^2$. This requires $G(0)$ to be known. In practice $G(0)$ may not be known at the receiver, so Figures 4, 5 and 6 serve to validate the correctness of our asymptotic theory, rather than to suggest the practical performance of the amplitude estimator. When SNR is large $G(0)$ is close to 1 and the bias of the amplitude estimator is small. However, $G(0)$ grows without bound as the

variance of the noise increases, so the bias is significant when SNR is small.

Figure 7 shows the MSE of the phase estimator when $M = 4$ and $L = 32, 256, 2048$ and the number of pilots is $|P| = \frac{L}{8}, L$. The figure depicts an interesting phenomenon. When $L = 2048$ and $|P| = \frac{L}{8} = 256$ the number of pilots symbols is the same as when $L = |P| = 256$. When the SNR is small (approximately less than 0 dB) the least squares estimator using the 256 pilots symbols and also the $2048 - 256 = 1792$ data symbols performs *worse* than the estimator that uses only the 256 pilots symbols. A similar phenomenon occurs when $L = 256$ and $|P| = \frac{L}{8} = 32$. This behaviour suggests modifying the objective function to give the pilots symbols more importance when the SNR is low. For example, rather than minimise (2) we could instead minimise a weighted version of it,

$$SS_\beta(a, \{d_i, i \in D\}) = \sum_{i \in P} |y_i - as_i|^2 + \beta \sum_{i \in D} |y_i - ad_i|^2,$$

where the weight β would be small when SNR is small and near 1 when SNR is large. Computing the \hat{a} that minimises SS_β can be achieved with only a minor modification to algorithm 1. Line 5 is modified to $g_i = \beta y_i e^{-j\hat{u}}$ and lines 7 and 17 are modified to $\hat{a} = \frac{1}{|P| + \beta|D|} Y$. For fixed β the asymptotic properties of this weighted estimator could be derived using the techniques we have developed in Sections IV, V and VI. This would enable a rigorous theory for selection of β at the receiver. One caveat is that the receiver would require knowledge about the noise distribution in order to advantageously choose β . We do not investigate this further here.

IX. CONCLUSION

We considered least squares estimators of carrier phase and amplitude from noisy communications signals that contain both pilot signals, known to the receiver, and data signals, unknown to the receiver. We focused on M -ary phase shift keying constellations. The least squares estimator can be computed in $O(L \log L)$ operations using a modification of an algorithm due to Mackenthun [7], and is the maximum likelihood estimator in the case that the noise is additive white and Gaussian.

We showed, under some reasonably general conditions on the distribution of the noise, that the phase estimator $\hat{\theta}$ is strongly consistent and asymptotically normally distributed. However, the amplitude estimator $\hat{\rho}_0$ is biased, and converges to $G(0)\rho_0$. This bias is large when the signal to noise ratio is small. It would be interesting to investigate methods for correcting this bias. A method for estimating $G(0)$ at the receiver appears to be required.

Monte Carlo simulations were used to assess the performance of the least squares estimator and also to validate our asymptotic theory. Interestingly, when the SNR is small, it is counterproductive to use the data symbols to estimate the phase (Figure 7). This suggests the use of a weighted objective function, which would be an interesting topic for future research.

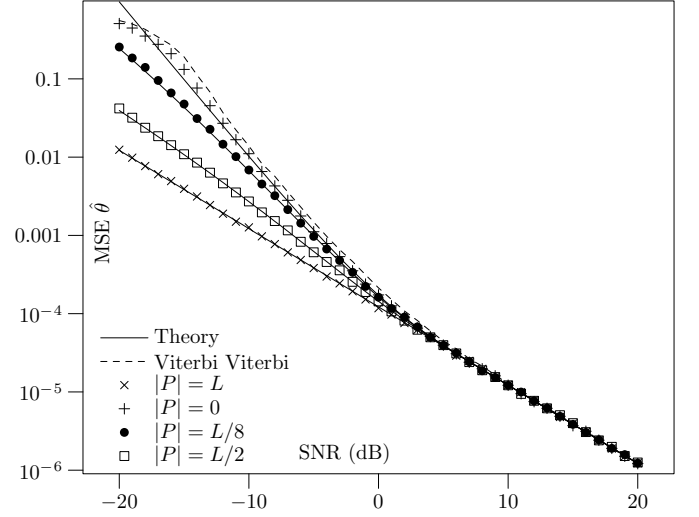


Fig. 1. Phase error versus SNR for BPSK with $L = 4096$.

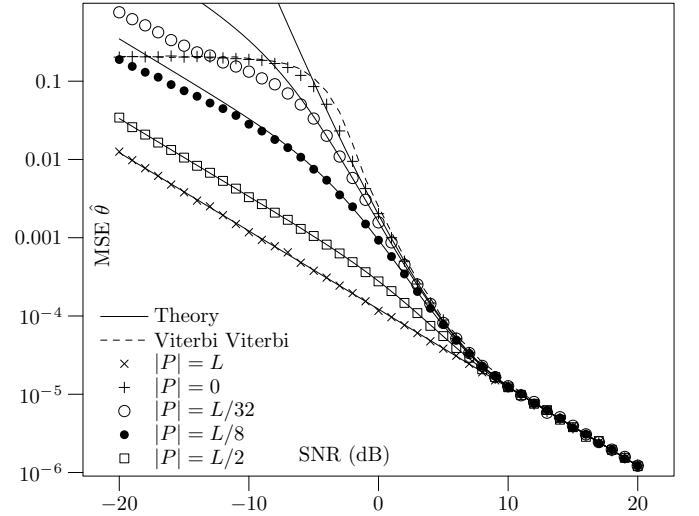


Fig. 2. Phase error versus SNR for QPSK with $L = 4096$.

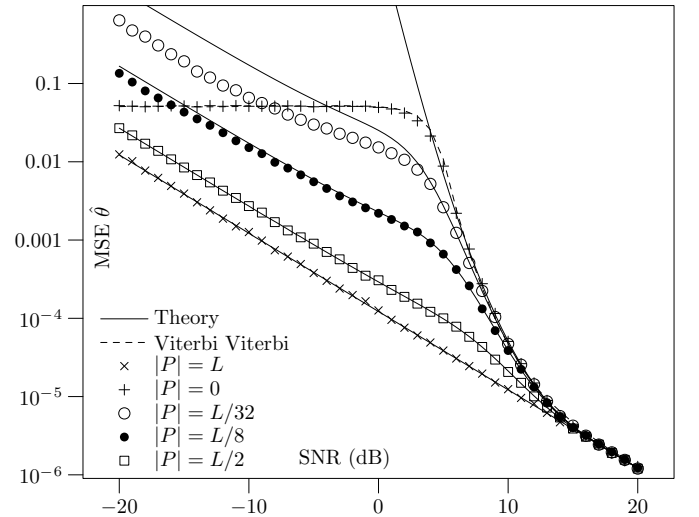


Fig. 3. Phase error versus SNR for 8-PSK with $L = 4096$.

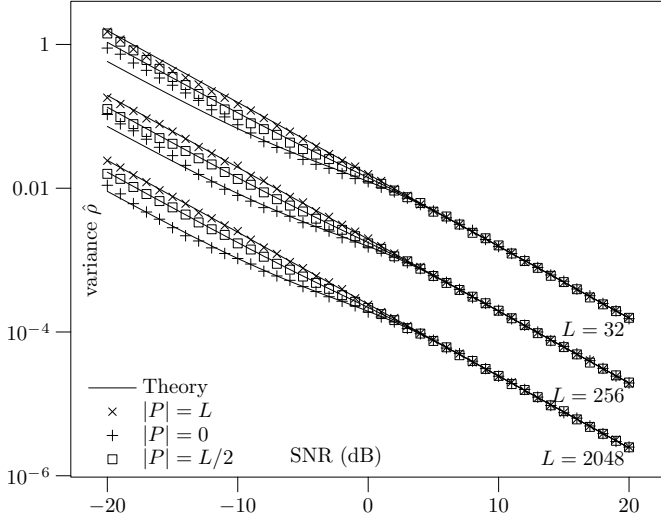


Fig. 4. Unbiased amplitude error versus SNR for BPSK.

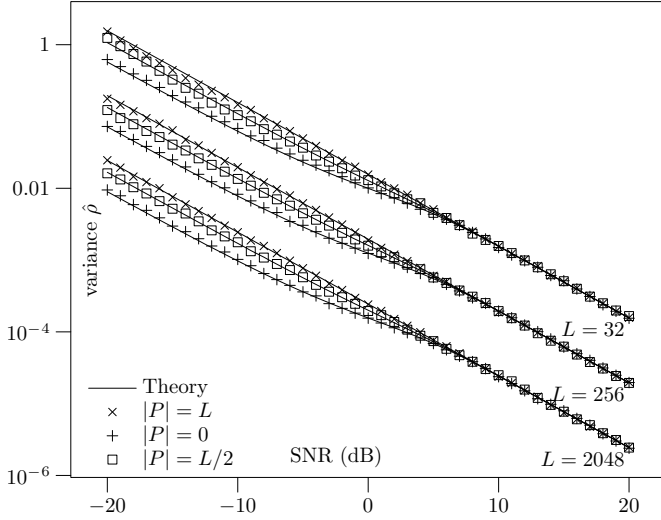


Fig. 5. Unbiased amplitude error versus SNR for QPSK.

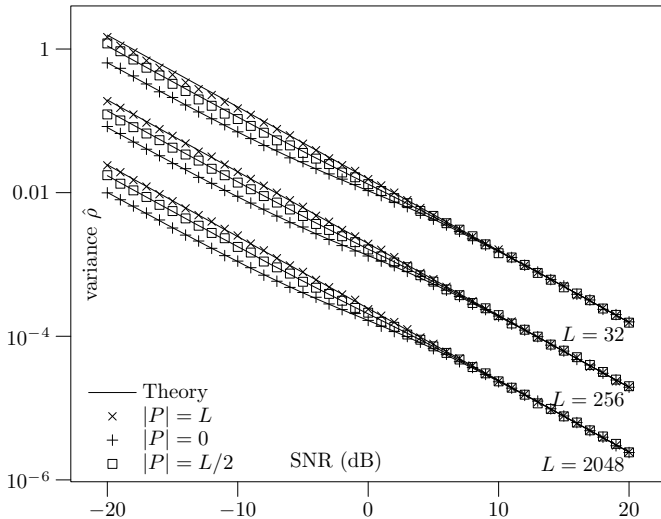


Fig. 6. Unbiased amplitude error versus SNR for 8-PSK.

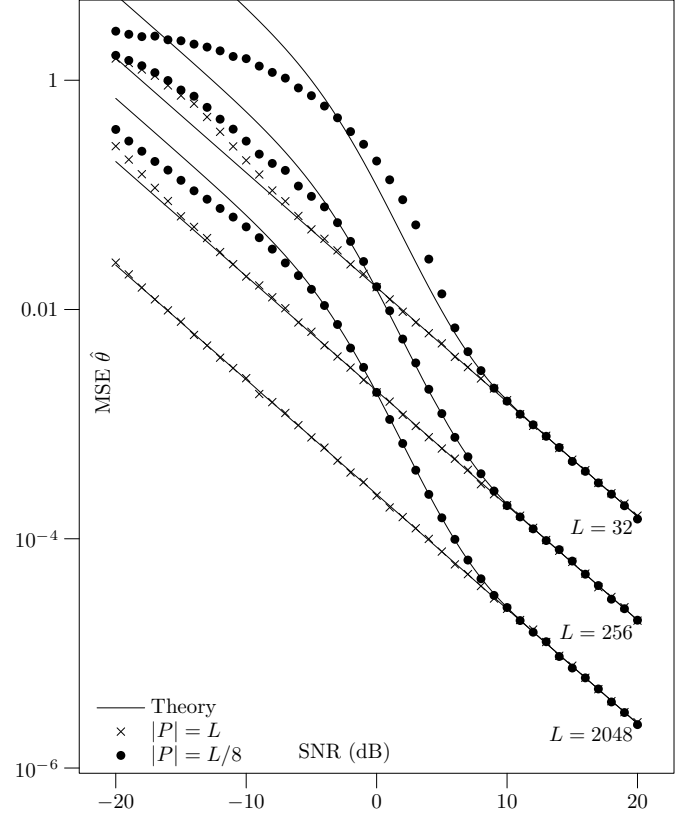


Fig. 7. Phase error versus SNR for QPSK.

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APPENDIX

A. Lemmas required for the proof of almost sure convergence (Theorem 1)

Lemma 1. $G(\hat{\lambda}_L) \rightarrow G(0)$ almost surely as $L \rightarrow \infty$.

Proof: Since $G(x)$ is uniquely maximised at $x = 0$,

$$0 \leq G(0) - G(\hat{\lambda}_L),$$

and since $\hat{\lambda}_L$ is the maximiser of $G_L(x)$,

$$0 \leq G_L(\hat{\lambda}_L) - G_L(0).$$

Thus,

$$\begin{aligned} 0 &\leq G(0) - G(\hat{\lambda}_L) \\ &\leq G(0) - G(\hat{\lambda}_L) + G_L(\hat{\lambda}_L) - G_L(0) \\ &\leq |G(0) - G_L(0)| + |G_L(\hat{\lambda}_L) - G(\hat{\lambda}_L)| \\ &\leq 2 \sup_{\lambda \in [-\pi, \pi]} |G_L(\lambda) - G(\lambda)|, \end{aligned}$$

and the last line converges almost surely to zero by Lemma 2. ■

Lemma 2. $\sup_{\lambda \in [-\pi, \pi]} |G_L(\lambda) - G(\lambda)| \rightarrow 0$ almost surely as $L \rightarrow \infty$.

Proof: Put $T_L(\lambda) = \mathbb{E}G_L(\lambda)$ and write

$$\begin{aligned} &\sup_{\lambda \in [-\pi, \pi]} |G_L(\lambda) - G(\lambda)| \\ &= \sup_{\lambda \in [-\pi, \pi]} |G_L(\lambda) - T_L(\lambda) + T_L(\lambda) - G(\lambda)| \\ &\leq \sup_{\lambda \in [-\pi, \pi]} |G_L(\lambda) - T_L(\lambda)| + \sup_{\lambda \in [-\pi, \pi]} |T_L(\lambda) - G(\lambda)|. \end{aligned}$$

Now,

$$\begin{aligned} T_L(\lambda) - G(\lambda) &= \left(\frac{|P|}{L} - p\right)h_1(\lambda) + \left(\frac{|D|}{L} - d\right)h_2(\lambda) \\ &= o(1)h_1(\lambda) + o(1)h_2(\lambda) \end{aligned}$$

Since

$$|h_1(\lambda)| = |\mathbb{E}R_1 \cos(\lambda + \Phi_1)| \leq \mathbb{E}R_1,$$

and

$$|h_2(\lambda)| = |\mathbb{E}R_1 \cos(\lambda + \Phi_1)| \leq \mathbb{E}R_1$$

for all $\lambda \in [-\pi, \pi]$, it follows that

$$\sup_{\lambda \in [-\pi, \pi]} |T_L(\lambda) - G(\lambda)| \leq o(1)\mathbb{E}R_1 \rightarrow 0$$

as $L \rightarrow \infty$. Lemma 3 shows that

$$\sup_{\lambda \in [-\pi, \pi]} |G_L(\lambda) - T_L(\lambda)| \rightarrow 0$$

almost surely as $L \rightarrow \infty$. ■

Lemma 3. Put $T_L(\lambda) = \mathbb{E}G_L(\lambda)$. Then

$$\sup_{\lambda \in [-\pi, \pi]} |G_L(\lambda) - T_L(\lambda)| \rightarrow 0$$

almost surely as $L \rightarrow \infty$.

Proof: Put $D_L(\lambda) = G_L(\lambda) - T_L(\lambda)$ and let

$$\lambda_n = \frac{2\pi}{N}(n-1) - \pi, \quad n = 1, \dots, N$$

be N points uniformly spaced on the interval $[-\pi, \pi]$. Let $L_n = [\lambda_n, \lambda_n + \frac{2\pi}{N})$ and observe that L_1, \dots, L_N partition $[-\pi, \pi]$. Now

$$\begin{aligned} &\sup_{\lambda \in [-\pi, \pi]} |D_L(\lambda)| \\ &= \sup_{n=1, \dots, N} \sup_{\lambda \in L_n} |D_L(\lambda) - D_L(\lambda_n) + D_L(\lambda_n)| \\ &\leq U_L + V_L, \end{aligned}$$

where

$$U_L = \sup_{n=1, \dots, N} |D_L(\lambda_n)| \quad \text{and}$$

$$V_L = \sup_{n=1, \dots, N} \sup_{\lambda \in L_n} |D_L(\lambda) - D_L(\lambda_n)|.$$

Lemma 4 shows that for any N and $\epsilon > 0$,

$$\Pr \left\{ \lim_{L \rightarrow \infty} U_L > \epsilon \right\} = 0,$$

that is, $U_L \rightarrow 0$ almost surely as $L \rightarrow \infty$. Lemma 5 shows that for any $\epsilon > 0$,

$$\Pr \left\{ \lim_{L \rightarrow \infty} V_L > \epsilon + \frac{4\pi}{N}\mathbb{E}R_i \right\} = 0.$$

If we choose N large enough that $4\pi\mathbb{E}R_i < \epsilon N$ then

$$\begin{aligned} &\Pr \left\{ \lim_{L \rightarrow \infty} \sup_{\lambda \in [-\pi, \pi]} |D_L(\lambda)| > 3\epsilon \right\} \\ &\leq \Pr \left\{ \lim_{L \rightarrow \infty} (U_L + V_L) > 3\epsilon \right\} \\ &\leq \Pr \left\{ \lim_{L \rightarrow \infty} (U_L + V_L) > \epsilon + \epsilon + \frac{4\pi}{N}\mathbb{E}R_i \right\} \\ &\leq \Pr \left\{ \lim_{L \rightarrow \infty} U_L > \epsilon \right\} + \Pr \left\{ \lim_{L \rightarrow \infty} V_L > \epsilon + \frac{4\pi}{N}\mathbb{E}R_i \right\} \\ &= 0. \end{aligned}$$

Thus $\sup_{\lambda \in [-\pi, \pi]} |D_L(\lambda)| \rightarrow 0$ almost surely as $L \rightarrow \infty$. ■

Lemma 4. For any $N > 0$, $U_L \rightarrow 0$ almost surely as $L \rightarrow \infty$ where U_L is defined in the proof of Lemma 3.

Proof: Put

$$Z_i(\lambda) = \begin{cases} R_i \cos(\lambda + \Phi_i), & i \in P \\ R_i \cos\langle \lambda + \Phi_i \rangle, & i \in D \end{cases}$$

so that

$$D_L(\lambda) = G_L(\lambda) - T_L(\lambda) = \frac{1}{L} \sum_{i \in P \cup D} (Z_i(\lambda) - \mathbb{E}Z_i(\lambda)).$$

Now $Z_1(\lambda_n), \dots, Z_L(\lambda_n)$ are independent with finite variance (because $\mathbb{E}R_i^2$ is finite), so for each $n = 1, \dots, N$,

$$|D_L(\lambda_n)| = \left| \frac{1}{L} \sum_{i \in P \cup D} (Z_i(\lambda_n) - \mathbb{E}Z_i(\lambda_n)) \right| \rightarrow 0$$

almost surely as $L \rightarrow \infty$ by Kolmogorov's strong law of large numbers [13]. Thus

$$U_L = \sup_{n=1, \dots, N} |D_L(\lambda_n)| \leq \sum_{n=1}^N |D_L(\lambda_n)| \rightarrow 0$$

almost surely to zero as $L \rightarrow \infty$. ■

Lemma 5. For any $\epsilon > 0$,

$$\Pr \left\{ \lim_{L \rightarrow \infty} V_L > \epsilon + \frac{4\pi}{N} \mathbb{E}R_i \right\} = 0.$$

Proof: Observe that

$$\begin{aligned} |D_L(\lambda) - D_L(\lambda_n)| &= |G_L(\lambda) - T_L(\lambda) - G_L(\lambda_n) + T_L(\lambda_n)| \\ &\leq |G_L(\lambda) - G_L(\lambda_n)| + |\mathbb{E}G_L(\lambda) - \mathbb{E}G_L(\lambda_n)| \\ &\leq |G_L(\lambda) - G_L(\lambda_n)| + \mathbb{E}|G_L(\lambda) - G_L(\lambda_n)|, \end{aligned}$$

the last line following from Jensen's inequality. Put

$$C_L = \sup_{n=1, \dots, N} \sup_{\lambda \in L_n} |G_L(\lambda) - G_L(\lambda_n)|,$$

so that

$$\begin{aligned} V_L &= \sup_{n=1, \dots, N} \sup_{\lambda \in L_n} |D_L(\lambda) - D_L(\lambda_n)| \\ &\leq C_L + \sup_{n=1, \dots, N} \sup_{\lambda \in L_n} \mathbb{E}|G_L(\lambda) - G_L(\lambda_n)| \\ &\leq C_L + \mathbb{E}C_L, \end{aligned}$$

where the last line follows because $\sup \mathbb{E}|\dots| \leq \mathbb{E} \sup |\dots|$. Lemma 6 shows that $\mathbb{E}C_L \leq \frac{2\pi}{N} \mathbb{E}R_1$ and also that

$$\Pr \left\{ \lim_{L \rightarrow \infty} C_L > \epsilon + \frac{2\pi}{N} \mathbb{E}R_1 \right\} = 0.$$

Thus,

$$\begin{aligned} \Pr \left\{ \lim_{L \rightarrow \infty} V_L > \epsilon + \frac{4\pi}{N} \mathbb{E}R_1 \right\} &\leq \Pr \left\{ \lim_{L \rightarrow \infty} (C_L + \mathbb{E}C_L) > \epsilon + \frac{4\pi}{N} \mathbb{E}R_1 \right\} \\ &\leq \Pr \left\{ \lim_{L \rightarrow \infty} C_L > \epsilon + \frac{2\pi}{N} \mathbb{E}R_1 \right\} = 0. \end{aligned}$$

Lemma 6. The following statements hold: ■

- 1) $\mathbb{E}C_L \leq \frac{2\pi}{N} \mathbb{E}R_1$ for all positive integers L ,
- 2) for any $\epsilon > 0$, $\Pr \left\{ \lim_{L \rightarrow \infty} C_L > \epsilon + \frac{2\pi}{N} \mathbb{E}R_1 \right\} = 0$.

Proof: If $\lambda \in L_n$, then $\lambda = \lambda_n + \delta$ with $\delta < \frac{2\pi}{N}$, and from Lemma 7,

$$\begin{aligned} |\cos(\lambda + \Phi_i) - \cos(\lambda_n + \Phi_i)| &\leq \frac{2\pi}{N}, \quad \text{and} \\ |\cos\langle \lambda + \Phi_i \rangle - \cos\langle \lambda_n + \Phi_i \rangle| &\leq \frac{2\pi}{N}. \end{aligned}$$

Because these results do not depend on n ,

$$\sup_{n=1, \dots, N} \sup_{\lambda \in L_n} |Z_i(\lambda) - Z_i(\lambda_n)| \leq R_i \frac{2\pi}{N}$$

for all $i \in P \cup D$. Also

$$\begin{aligned} C_L &= \sup_{n=1, \dots, N} \sup_{\lambda \in L_n} \left| \frac{1}{L} \sum_{i \in P \cup D} Z_i(\lambda) - Z_i(\lambda_n) \right| \\ &\leq \frac{1}{L} \sum_{i \in P \cup D} \sup_{n=1, \dots, N} \sup_{\lambda \in L_n} |Z_i(\lambda) - Z_i(\lambda_n)| \\ &\leq \frac{2\pi}{NL} \sum_{i \in P \cup D} R_i. \end{aligned}$$

Thus,

$$\mathbb{E}C_L \leq \mathbb{E} \frac{2\pi}{NL} \sum_{i \in P \cup D} R_i = \frac{2\pi}{N} \mathbb{E}R_1$$

and the first statement holds. Now,

$$\frac{2\pi}{NL} \sum_{i \in P \cup D} R_i \rightarrow \frac{2\pi}{N} \mathbb{E}R_1$$

almost surely as $L \rightarrow \infty$ by the strong law of large numbers, and so, for any $\epsilon > 0$,

$$\begin{aligned} \Pr \left\{ \lim_{L \rightarrow \infty} C_L > \epsilon + \frac{2\pi}{N} \mathbb{E}R_1 \right\} &\leq \Pr \left\{ \lim_{L \rightarrow \infty} \frac{2\pi}{NL} \sum_{i \in P \cup D} R_i > \epsilon + \frac{2\pi}{N} \mathbb{E}R_1 \right\} = 0. \end{aligned}$$

Lemma 7. Let x and δ be real numbers. Then

$$\begin{aligned} |\cos(x + \delta) - \cos(x)| &\leq |\delta|, \quad \text{and} \\ |\cos\langle x + \delta \rangle - \cos\langle x \rangle| &\leq |\delta|. \end{aligned}$$

Proof: Both $\cos(x)$ and $\cos\langle x \rangle$ are Lipschitz continuous functions from \mathbb{R} to \mathbb{R} with constant $K = 1$. That is, for any x and y in \mathbb{R} ,

$$\begin{aligned} |\cos(y) - \cos(x)| &\leq K|x - y| = |x - y|, \quad \text{and} \\ |\cos\langle y \rangle - \cos\langle x \rangle| &\leq K|x - y| = |x - y|. \end{aligned}$$

The lemma follows by putting $y = x + \delta$. ■

Lemma 8. $G_L(\hat{\lambda}_L) \rightarrow G(0)$ almost surely as $L \rightarrow \infty$.

Proof: By the triangle inequality,

$$|G_L(\hat{\lambda}_L) - G(0)| \leq |G_L(\hat{\lambda}_L) - G(\hat{\lambda}_L)| + |G(\hat{\lambda}_L) - G(0)|.$$

Now $|G_L(\hat{\lambda}_L) - G(\hat{\lambda}_L)| \rightarrow 0$ as $L \rightarrow \infty$ as a result of Lemma 2, and $|G(\hat{\lambda}_L) - G(0)| \rightarrow 0$ almost surely as $L \rightarrow \infty$ because G is continuous and $\hat{\lambda}_L \rightarrow 0$ almost surely as $L \rightarrow \infty$. ■

B. Lemmas required for the proof of asymptotic normality (Theorem 2)

Lemma 9. *The derivative of G_L exists, and is equal to zero, at $\hat{\lambda}_L$. That is,*

$$G'_L(\hat{\lambda}_L) = \frac{dG_L}{d\lambda}(\hat{\lambda}_L) = 0.$$

Proof: Observe that

$$G_L(\lambda) = \frac{1}{L} \sum_{i \in P} R_i \cos(\lambda + \Phi_i) + \frac{1}{L} \sum_{i \in D} R_i \cos(\lambda + \Phi_i)$$

is differentiable everywhere except when $\langle \lambda + \Phi_i \rangle = -\frac{\pi}{M}$ for any $i \in D$ with $R_i > 0$. Let q_i be the smallest number from the interval $[-\frac{\pi}{M}, 0]$ such that

$$\sin(q_i) > -\frac{\rho_0 |\eta|^2 R_i}{4L\hat{\rho} \sin(\pi/M)}$$

where $\eta = e^{-j2\pi/M} - 1$. Observe that $q_i < 0$ when $R_i > 0$. Lemma 10 shows that

$$|\langle \hat{\lambda}_L + \Phi_i \rangle| \leq \frac{\pi}{M} + q_i$$

for all $i \in D$. Thus, $\langle \hat{\lambda}_L + \Phi_i \rangle \neq -\frac{\pi}{M}$ for $i \in D$ such that $R_i > 0$ and therefore G_L is differentiable at $\hat{\lambda}_L$. That $G'_L(\hat{\lambda}_L) = 0$ follows since $\hat{\lambda}_L$ is a maximiser of G_L . ■

Lemma 10. *Let q_i be defined as in Lemma 9. Then $|\langle \hat{\lambda}_L + \Phi_i \rangle| \leq \frac{\pi}{M} + q_i$ for all $i \in D$.*

Proof: Recall that $\{\hat{d}_i = \hat{d}_i(\hat{\theta}), i \in D\}$ defined in (8) are the minimisers of the function

$$SS(\{d_i, i \in D\}) = A - \frac{1}{L} |Y(\{d_i, i \in D\})|^2,$$

defined in (6). The proof now proceeds by contradiction. Assume that

$$\langle \hat{\lambda}_L + \Phi_k \rangle > \frac{\pi}{M} + q_k \quad (23)$$

for some $k \in D$. Recalling the notation e_k defined in Section II, put $r_i = \hat{d}_i e_k$. We will show that

$$SS(\{r_i, i \in D\}) < SS(\{\hat{d}_i, i \in D\}),$$

violating the fact that $\{\hat{d}_i, i \in D\}$ are minimisers of SS . First observe that,

$$Y(\{r_i, i \in D\}) = \sum_{i \in P} y_i p_i^* + \sum_{i \in D} y_i \hat{r}_i^* = \hat{Y} + \eta y_k \hat{d}_k^*,$$

where $\eta = e^{-j2\pi/M} - 1$ and $\hat{Y} = Y(\{\hat{d}_i, i \in D\})$. Now,

$$\begin{aligned} SS(\{r_i, i \in D\}) &= A - \frac{1}{L} |Y(\{r_i, i \in D\})|^2 \\ &= A - \frac{1}{L} |\hat{Y} + \eta y_k \hat{d}_k^*|^2 \\ &= A - \frac{1}{L} |\hat{Y}|^2 - \frac{2}{L} \Re(\eta \hat{Y}^* y_k \hat{d}_k^*) - \frac{1}{L} |\eta y_k|^2 \\ &= SS(\{\hat{d}_i, i \in D\}) - C, \end{aligned}$$

where

$$C = \frac{2}{L} \Re(\eta \hat{Y}^* y_k \hat{d}_k^*) + \frac{1}{L} |\eta y_k|^2.$$

Now $\frac{1}{L} \hat{Y} = \hat{a} = \hat{\rho} e^{j\hat{\theta}}$ from (5) and using (19),

$$\frac{1}{L} \hat{Y}^* y_k \hat{d}_k^* = \hat{\rho} \rho_0 R_k e^{j\langle \hat{\lambda}_L + \Phi_k \rangle},$$

so that

$$C = 2\hat{\rho} \rho_0 R_k \Re\left(\eta e^{j\langle \hat{\lambda}_L + \Phi_k \rangle}\right) + \frac{1}{L} |\eta|^2 \rho_0^2 R_k^2. \quad (24)$$

Let $v = \langle \hat{\lambda}_L + \Phi_k \rangle - \frac{\pi}{M}$ so that

$$\begin{aligned} \eta e^{j\langle \hat{\lambda}_L + \Phi_k \rangle} &= (e^{-j2\pi/M} - 1) e^{j\pi/M} e^{jv} \\ &= (e^{-j\pi/M} - e^{j\pi/M}) e^{jv} \\ &= -2j \sin\left(\frac{\pi}{M}\right) e^{jv}, \end{aligned}$$

and

$$\Re\left(\eta e^{j\langle \hat{\lambda}_L + \Phi_k \rangle}\right) = 2 \sin\left(\frac{\pi}{M}\right) \sin(v).$$

Because we assumed (23), it follows that $0 > v > q_k$ and, from the definition of q_k ,

$$-\frac{\rho_0 |\eta|^2 R_k}{4L\hat{\rho} \sin(\pi/M)} < \sin(v) < 0.$$

Substituting this into (24) gives $C > 0$, but then

$$SS(\{r_i, i \in D\}) < SS(\{\hat{d}_i, i \in D\}),$$

violating the fact that $\{\hat{d}_i, i \in D\}$ are minimisers of SS . So (23) is false.

To show that $\langle \hat{\lambda}_L + \Phi_k \rangle \geq -\frac{\pi}{M} - q_k$ we use contradiction again. Assume that $\langle \hat{\lambda}_L + \Phi_k \rangle < -\frac{\pi}{M} - q_k$. Recalling the notation e_k^* defined in Section II, put $r_i = \hat{d}_i e_k^*$. Now an analogous argument can be used to show that $SS(\{r_i, i \in D\}) < SS(\{\hat{d}_i, i \in D\})$ again. ■

Lemma 11. *Let $Q_L(\lambda) = \mathbb{E}R_L(\lambda) - \mathbb{E}R_L(0)$ where the function R_L is defined in (22). We have*

$$\sqrt{L}Q_L(\hat{\lambda}_L) = \sqrt{L}\hat{\lambda}_L(p + Hd + o_P(1))$$

where p, d and H are defined in the statements of Theorems 1 and 2.

Proof: We have

$$Q_L(\lambda) = \mathbb{E}R_L(\lambda) - \mathbb{E}R_L(0) = \frac{|P|}{L} k_1(\lambda) + \frac{|D|}{L} k_2(\lambda)$$

where

$$\begin{aligned} k_1(\lambda) &= \mathbb{E}R_1(\sin(\lambda + \Phi_1) - \sin(\Phi_1)), \quad \text{and} \\ k_2(\lambda) &= \mathbb{E}R_1(\sin(\lambda + \Phi_1) - \sin(\Phi_1)). \end{aligned} \quad (25)$$

Lemma 12 shows that $k_1(\hat{\lambda}_L) = \hat{\lambda}_L(1 + o_P(1))$ and Lemma 13 shows that $k_2(\hat{\lambda}_L) = \hat{\lambda}_L(H + o_P(1))$ and so

$$\begin{aligned} Q_L(\hat{\lambda}_L) &= \frac{|P|}{L} \hat{\lambda}_L(1 + o_P(1)) + \frac{|D|}{L} \hat{\lambda}_L(H + o_P(1)) \\ &= \hat{\lambda}_L \left(\frac{|P|}{L} + \frac{|D|}{L} H + o_P(1) \right) \\ &= \hat{\lambda}_L(p + dH + o_P(1)), \end{aligned}$$

since $\frac{|P|}{L} \rightarrow p$ and $\frac{|D|}{L} \rightarrow d$ as $L \rightarrow \infty$. The lemma follows by multiplying both sides of the above equation by \sqrt{L} . ■

Lemma 12. *Put*

$$q_1(\lambda) + jk_1(\lambda) = \mathbb{E}[R_1 e^{j(\lambda + \Phi_1)} - R_1 e^{j\Phi_1}].$$

We have $q_1(\lambda) = \hat{\lambda}_L o_P(1)$ and $k_1(\hat{\lambda}_L) = \hat{\lambda}_L(1 + o_P(1))$.

Proof: We have

$$q_1(\lambda) + jk_1(\lambda) = (e^{j\lambda} - 1)\mathbb{E}R_1 e^{j\Phi_1} = e^{j\lambda} - 1$$

since the mean of $R_1 e^{j\Phi_1}$ is 1. By a first order expansion about $\lambda = 0$ we obtain

$$q_1(\lambda) + jk_1(\lambda) = \lambda(j + O(\lambda))$$

Since $\hat{\lambda}_L$ converges almost surely to zero as $L \rightarrow \infty$ it follows that $O(\hat{\lambda}_L) = o_P(1)$. Thus

$$q_1(\hat{\lambda}_L) + jk_1(\hat{\lambda}_L) = \hat{\lambda}_L(j + o_P(1))$$

and the lemma follow by taking real and imaginary parts. ■

Lemma 13. *Put*

$$q_2(\lambda) + jk_2(\lambda) = \mathbb{E}[R_1 e^{j(\lambda + \Phi_1)} - R_1 e^{j\langle \Phi_1 \rangle}].$$

We have $q_2(\hat{\lambda}_L) = \hat{\lambda}_L o_P(1)$ and $k_2(\hat{\lambda}_L) = \hat{\lambda}_L(H + o_P(1))$ where H is defined in the statement of Theorem 2.

Proof: Because $\hat{\lambda}_L \rightarrow 0$ almost surely as $L \rightarrow \infty$, it is only the behaviour of $q_2(\lambda)$ and $k_2(\lambda)$ around $\lambda = 0$ that is relevant. We will examine $q_2(\lambda)$ and $k_2(\lambda)$ for $0 \leq \lambda < \frac{\pi}{M}$. An analogous argument follows when $-\frac{\pi}{M} < \lambda < 0$. To keep our notation clean put

$$\psi_k = \frac{2\pi}{M}k + \frac{\pi}{M}$$

with $k \in \mathbb{Z}$. When $\Phi_1 \in [\psi_{k-1}, \psi_k - \lambda)$,

$$\begin{aligned} e^{j(\lambda + \Phi_1)} - e^{j\langle \Phi_1 \rangle} &= e^{j(\lambda + \Phi_1 - \frac{2\pi}{M}k)} - e^{j(\Phi_1 - \frac{2\pi}{M}k)} \\ &= (e^{j\lambda} - 1)e^{j(\Phi_1 - \frac{2\pi}{M}k)} \\ &= (e^{j\lambda} - 1)e^{j\langle \Phi_1 \rangle}, \end{aligned}$$

and when $\Phi_1 \in [\psi_k - \lambda, \psi_k)$,

$$\begin{aligned} e^{j(\lambda + \Phi_1)} - e^{j\langle \Phi_1 \rangle} &= e^{j(\lambda + \Phi_1 - \frac{2\pi}{M}k - \frac{2\pi}{M})} - e^{j(\Phi_1 - \frac{2\pi}{M}k)} \\ &= (e^{j(\lambda - \frac{2\pi}{M})} - 1)e^{j\langle \Phi_1 \rangle} \\ &= (e^{j\lambda} - 1)e^{j\langle \Phi_1 \rangle} + e^{j\lambda}(e^{-j\frac{2\pi}{M}} - 1)e^{j\langle \Phi_1 \rangle}. \end{aligned}$$

Thus, when $\Phi_1 \in [\psi_{k-1}, \psi_k)$,

$$\begin{aligned} e^{j(\lambda + \Phi_1)} - e^{j\langle \Phi_1 \rangle} &= (e^{j\lambda} - 1)e^{j\langle \Phi_1 \rangle} \\ &\quad + e^{j\lambda}(e^{-j\frac{2\pi}{M}} - 1)e^{j\langle \Phi_1 \rangle}\chi_k(\Phi_1, \lambda) \end{aligned}$$

where

$$\chi_k(\Phi_1, \lambda) = \begin{cases} 1, & \Phi_1 \in [\psi_k - \lambda, \psi_k) \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} q_2(\lambda) + jk_2(\lambda) &= \mathbb{E}[R_1 e^{j(\lambda + \Phi_1)} - R_1 e^{j\langle \Phi_1 \rangle}] \\ &= (e^{j\lambda} - 1)h_2(0) + e^{j\lambda}B(\lambda) \end{aligned} \quad (26)$$

since $\mathbb{E}R_1 e^{j\langle \Phi_1 \rangle} = h_2(0) + \mathbb{E}R_1 \sin \langle \Phi_1 \rangle = h_2(0)$ as a result of Lemma 15 and where

$$B(\lambda) = (e^{-j\frac{2\pi}{M}} - 1)\mathbb{E}R_1 e^{j\langle \Phi_1 \rangle}\chi(\Phi_1, \lambda) \quad (27)$$

and $\chi(\Phi_1, \lambda) = \sum_{k \in \mathbb{Z}} \chi_k(\Phi_1, \lambda)$.

Now $e^{j\hat{\lambda}_L} = 1 + o_P(1)$ and $e^{j\hat{\lambda}_L} - 1 = \hat{\lambda}_L(j + o_P(1))$ by the argument in Lemma 12. Also

$$B(\hat{\lambda}_L) = -\hat{\lambda}_L \left(2j \sin\left(\frac{\pi}{M}\right) \sum_{k=0}^{M-1} g(\psi_k) + o_P(1) \right)$$

by Lemma 14. Combining these results into (26) we obtain

$$\begin{aligned} q_2(\lambda) + jk_2(\hat{\lambda}_L) &= \hat{\lambda}_L \left(jh_2(0) - 2j \sin\left(\frac{\pi}{M}\right) \sum_{k=0}^{M-1} g(\psi_k) + o_P(1) \right) \\ &= \hat{\lambda}_L(jH + o_P(1)) \end{aligned}$$

and the lemma follows by taking real and imaginary parts. ■

Lemma 14. *With $B(\lambda)$ defined in (27) we have*

$$B(\hat{\lambda}_L) = -\hat{\lambda}_L \left(2j \sin\left(\frac{\pi}{M}\right) \sum_{k=0}^{M-1} g(\psi_k) + o_P(1) \right)$$

Proof: Put $A(\lambda) = \mathbb{E}R_1 e^{j\langle \Phi_1 \rangle} \chi(\Phi_1, \lambda)$. Recalling that $f(r, \phi)$ is the joint pdf of R_1 and Φ_1 we have

$$\begin{aligned} A(\lambda) &= \int_0^{2\pi} \int_0^\infty r f(r, \phi) e^{j\langle \phi \rangle} \chi(\phi, \lambda) dr d\phi \\ &= \sum_{k \in \mathbb{Z}} \int_0^{2\pi} g(\phi) e^{j\langle \phi \rangle} \chi_k(\phi, \lambda) d\phi \\ &= \sum_{k=0}^{M-1} \int_{\psi_k - \lambda}^{\psi_k} g(\phi) e^{j\langle \phi \rangle} d\phi, \end{aligned}$$

the last line because the $\chi_k(\phi, \lambda)$ terms inside the integral are zero for all $\phi \in [0, 2\pi]$ when $k \notin \{0, \dots, M-1\}$. Observe that $\langle \phi \rangle \rightarrow \frac{\pi}{M}$ as ϕ approaches ψ_k from below. Because $g(\psi_k)$ is continuous at ψ_k for each $k = 0, \dots, M-1$ (by assumption in Theorem 2) we have

$$\frac{1}{\lambda} A(\lambda) \rightarrow \sum_{k=0}^{M-1} g(\psi_k) e^{j\frac{\pi}{M}}$$

as λ approaches zero from above. We are only interested in the limit from above because we are working under the assumption that $0 \leq \lambda < \frac{\pi}{M}$ (see the proof of Lemma 13). The analogous argument when $-\frac{\pi}{M} \leq \lambda < 0$ would involve limits as λ approaches zero from below. Thus

$$A(\hat{\lambda}_L) = \hat{\lambda}_L \left(e^{j\frac{\pi}{M}} \sum_{k=0}^{M-1} g(\psi_k) + o_P(1) \right)$$

and the lemma follows since $(e^{-j\frac{2\pi}{M}} - 1)e^{j\frac{\pi}{M}} = -2j \sin(\frac{\pi}{M})$ and $B(\lambda) = (e^{-j\frac{2\pi}{M}} - 1)A(\lambda)$. ■

Lemma 15. $\mathbb{E}R_1 \sin \langle \Phi_1 \rangle = 0$.

Proof: Recalling that $f(r, \phi)$ is the joint pdf of R_1 and Φ_1 we have

$$\begin{aligned} \mathbb{E}R_1 \sin \langle \Phi_1 \rangle &= \int_0^{2\pi} \int_0^\infty r \sin \langle \phi \rangle f(r, \phi) dr d\phi \\ &= \int_{-\pi}^\pi \sin \langle \phi \rangle g(\phi) d\phi. \end{aligned}$$

The proof is immediate since $g(\phi)$ is even and $\sin \langle \phi \rangle$ is odd. ■

Lemma 16. *Let $Q_L(\lambda) = \mathbb{E}R_L(\lambda) - \mathbb{E}R_L(0)$ where the function R_L is defined in (22). Then,*

$$\sqrt{L}(R_L(\hat{\lambda}_L) - Q_L(\hat{\lambda}_L)) = \sqrt{L}R_L(0) + o_P(1).$$

Proof: Write

$$\sqrt{L}(R_L(\hat{\lambda}_L) - Q_L(\hat{\lambda}_L)) = W_L(\hat{\lambda}_L) + \sqrt{L}R_L(0)$$

where

$$W_L(\lambda) = \sqrt{L}(R_L(\lambda) - Q_L(\lambda) - R_L(0)) \quad (28)$$

is what is called an *empirical process* indexed by λ [16–19]. Techniques from this literature can be used to show that for any $\delta > 0$ and $\nu > 0$, there exists $\epsilon > 0$ such that

$$\Pr \left\{ \sup_{|\lambda| < \epsilon} |W_L(\lambda)| > \delta \right\} < \nu$$

for all positive integers L . This type of result is typically called *tightness* or *asymptotic continuity* [18, 19, 21]. We omit the proof which follows in a straightforward, but lengthy manner using an argument called *symmetrisation* followed by an argument called *chaining* [18, 19].

Since $\hat{\lambda}_L$ converges almost surely to zero, it follows that for any $\epsilon > 0$,

$$\lim_{L \rightarrow \infty} \Pr \left\{ |\hat{\lambda}_L| \geq \epsilon \right\} = 0$$

and therefore, for any $\nu > 0$, $\Pr\{|\hat{\lambda}_L| \geq \epsilon\} < \nu$ for all sufficiently large L . Now

$$\begin{aligned} & \Pr \left\{ |W_L(\hat{\lambda}_L)| > \delta \right\} \\ &= \Pr \left\{ |W_L(\hat{\lambda}_L)| > \delta \text{ and } |\hat{\lambda}_L| < \epsilon \right\} \\ &\quad + \Pr \left\{ |W_L(\hat{\lambda}_L)| > \delta \text{ and } |\hat{\lambda}_L| \geq \epsilon \right\} \\ &\leq \Pr \left\{ \sup_{|\lambda| < \epsilon} |W_L(\lambda)| > \delta \right\} + \Pr \left\{ |\hat{\lambda}_L| \geq \epsilon \right\} \\ &\leq 2\nu, \end{aligned}$$

for all sufficiently large L . Since ν and δ can be chosen arbitrarily small, it follows that $W_L(\hat{\lambda}_L)$ converges in probability to zero as $N \rightarrow \infty$. ■

Lemma 17. *The distribution of $\sqrt{L}R_L(0)$ converges to the normal with zero mean and variance $pA_1 + dA_2$ as $L \rightarrow \infty$.*

Proof: Observe that $\sqrt{L}R_L(0) = C_L + D_L$ where

$$C_L = \frac{1}{\sqrt{L}} \sum_{i \in P} R_i \sin(\Phi_i), \quad D_L = \frac{1}{\sqrt{L}} \sum_{i \in D} R_i \sin(\Phi_i).$$

From the standard central limit theorem the distribution of C_L converges to the normal with mean $\sqrt{p}\mathbb{E}R_1 \sin(\Phi_1) = 0$ as a result of (16), and variance

$$pA_1 = p\mathbb{E}R_1^2 \sin^2(\Phi_1).$$

Similarly, the distribution of D_L converges to the normal with mean $\sqrt{d}\mathbb{E}R_1 \sin \langle \Phi_1 \rangle = 0$ as a result of Lemma 15, and variance

$$dA_2 = d\mathbb{E}R_1^2 \sin^2 \langle \Phi_1 \rangle.$$

The lemma holds because C_L and D_L are independent. ■

Lemma 18. *Let $T_L(\lambda) = \mathbb{E}G_L(\lambda)$. We have*

$$\sqrt{L}(G_L(\hat{\lambda}_L) - T_L(\hat{\lambda}_L)) = X_L + o_P(1),$$

where $X_L = \sqrt{L}(G_L(0) - T_L(0))$.

Proof: Write

$$\sqrt{L}(G_L(\hat{\lambda}_L) - T_L(\hat{\lambda}_L)) = Y_L(\hat{\lambda}_L) + X_L$$

where

$$Y_L(\lambda) = \sqrt{L}(G_L(\lambda) - T_L(\lambda)) - X_L. \quad (29)$$

is an empirical process indexed by λ , similar to W_L from (28). As with W_L results from the literature on empirical processes can be used to show that for any $\delta > 0$ and $\nu > 0$, there exists $\epsilon > 0$ such that

$$\Pr \left\{ \sup_{|\lambda| < \epsilon} |Y_L(\lambda)| > \delta \right\} < \nu.$$

The proof now follows by an argument analogous to that in Lemma 16. ■

Lemma 19. $\sqrt{L}(T_L(\hat{\lambda}_L) - G(0)) = o_P(1)$.

Proof: The argument is similar to that used in Lemma 11. First observe that

$$\begin{aligned} T_L(\lambda) &= \frac{|P|}{L} \mathbb{E}R_1 \cos(\lambda + \Phi_1) + \frac{|D|}{L} \mathbb{E}R_1 \cos \langle \lambda + \Phi_1 \rangle \\ &= (p + o(L^{-1/2})) \mathbb{E}R_1 \cos(\lambda + \Phi_1) \\ &\quad + (d + o(L^{-1/2})) \mathbb{E}R_1 \cos \langle \lambda + \Phi_1 \rangle, \end{aligned}$$

and because $\frac{|P|}{L} = p + o(L^{-1/2})$ and $\frac{|D|}{L} = d + o(L^{-1/2})$ (by assumption in Theorem 2), we have

$$\sqrt{L}(T_L(\lambda) - G(0)) = \sqrt{L}pq_1(\lambda) + \sqrt{L}dq_2(\lambda) + o(1),$$

where

$$\begin{aligned} q_1(\lambda) &= \mathbb{E}R_1 (\cos(\lambda + \Phi_1) - \cos(\Phi_1)) \quad \text{and} \\ q_2(\lambda) &= \mathbb{E}R_1 (\cos \langle \lambda + \Phi_1 \rangle - \cos \langle \Phi_1 \rangle). \end{aligned} \quad (30)$$

Lemma 12 shows that $q_1(\hat{\lambda}_L) = \hat{\lambda}_L o_P(1)$ and Lemma 13 shows that $q_2(\hat{\lambda}_L) = \hat{\lambda}_L o_P(1)$ and so

$$\sqrt{L}(T_L(\lambda) - G(0)) = \sqrt{L}\hat{\lambda}_L o_P(1) + o(1).$$

The lemma follows since $\sqrt{L}\hat{\lambda}_L$ converges in distribution. ■

Lemma 20. *The distribution of*

$$X_L = \sqrt{L}(G_L(0) - T_L(0)) = \sqrt{L}(G_L(0) - \mathbb{E}G_L(0))$$

converges, as $L \rightarrow \infty$, to the normal with zero mean and covariance $pB_1 + dB_2$.

Proof: Observe that $X_L = C'_L + D'_L$ where

$$C'_L = \frac{1}{\sqrt{L}} \sum_{i \in P} (R_i \cos(\Phi_i) - h_1(0)),$$

$$D'_L = \frac{1}{\sqrt{L}} \sum_{i \in D} (R_i \cos \langle \Phi_i \rangle - h_2(0)),$$

where h_1 and h_2 are defined in the statement of Theorem 1. From the standard central limit theorem the distribution of C'_L converges to the normal with zero mean and variance

$$pB_1 = p\mathbb{E}R_1^2 \cos^2(\Phi_1) - ph_1^2(0) = p\mathbb{E}R_1^2 \cos^2(\Phi_1) - p$$

since $h_1(0) = 1$. Similarly the distribution of D'_L converges to the normal with zero mean and variance

$$dB_2 = d\mathbb{E}R_1^2 \cos^2 \langle \Phi_1 \rangle - dh_2^2(0).$$

The lemma follows since C'_L and D'_L are independent. ■